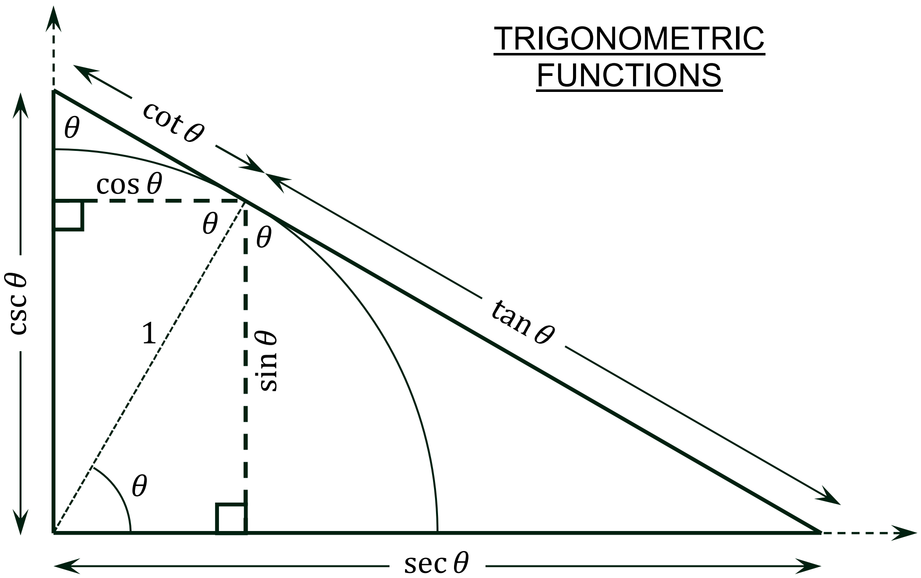
TRIGONOMETRIC FUNCTIONS



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$

RECIPROCAL IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$	$\csc(\theta) = \frac{1}{\sin(\theta)}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$	$\sec(\theta) = \frac{1}{\cos(\theta)}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$	$\cot(\theta) = \frac{1}{\tan(\theta)}$

RATIO IDENTITIES

$\sin(\theta) = \frac{O}{H}$	$\csc(\theta) = \frac{H}{O}$
$\cos(\theta) = \frac{A}{H}$	$\sec(\theta) = \frac{H}{A}$
$\tan(\theta) = \frac{O}{A}$	$\cot(\theta) = \frac{A}{O}$
$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$
$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$	$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$

PYTHAGOREAN IDENTITIES

$$\cos(\theta) = \frac{A}{H}$$

$$\sin(\theta) = \frac{O}{H}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

SYMMETRY IDENTITIES

The graph of $\sin(\theta)$ is symmetric about the origin, therefore…

$$\sin(-\theta) = -\sin(\theta)$$

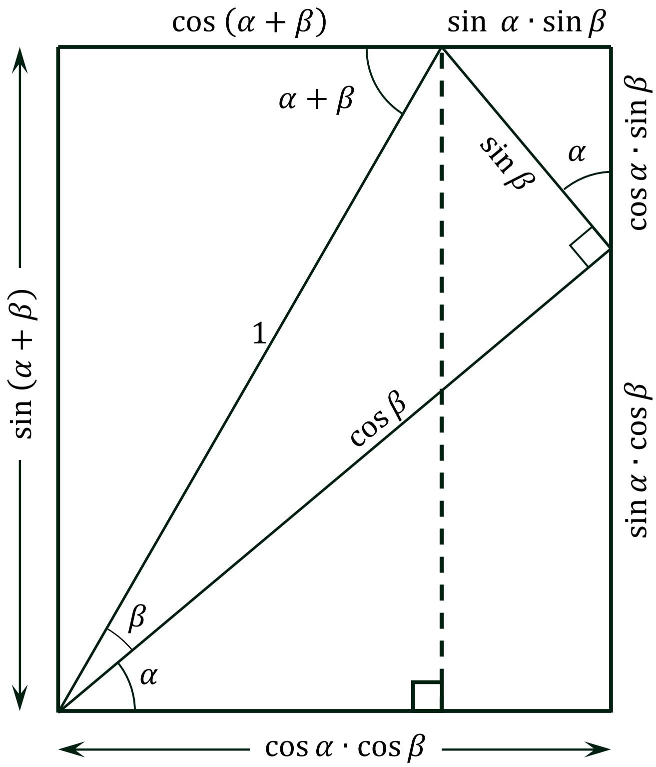
The graph of $\cos(\theta)$ is symmetric about the y-axis, therefore…

$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin, therefore…

$$\tan(-\theta) = -\tan(\theta)$$

SIN and COS of SUMS (and DIFFERENCES)



$$\boxed{\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta}$$

$$\sin(\alpha-\beta)=\sin\big(\alpha+(-\beta)\big)$$
$$=\sin\alpha\cdot\cos(-\beta)+\cos\alpha\cdot\sin(-\beta)$$

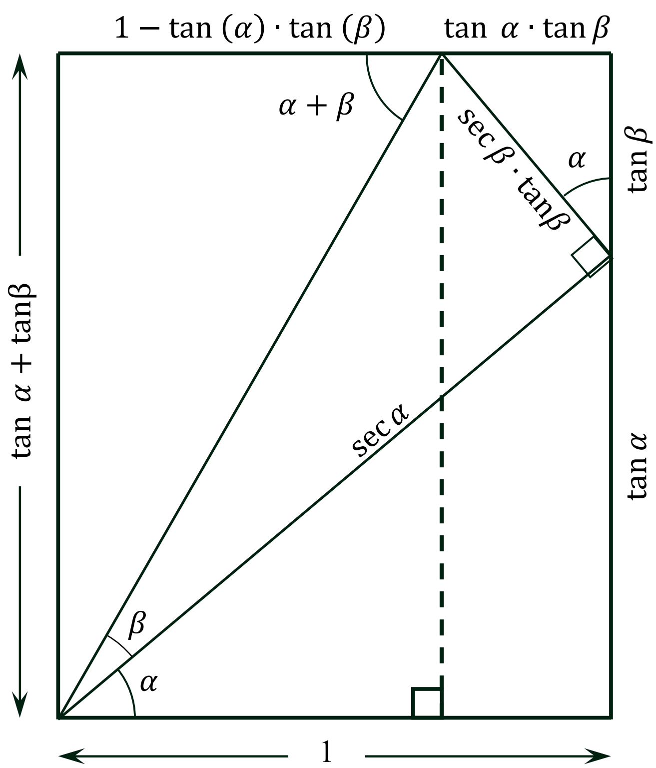
$$\boxed{\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta}$$

$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta}$$

$$\cos(\alpha-\beta)=\cos\big(\alpha+(-\beta)\big)$$
$$=\cos\alpha\cdot\cos(-\beta)-\sin\alpha\cdot\sin(-\beta)$$

$$\boxed{\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta}$$

TAN of SUMS (and DIFFERENCES)



$$\boxed{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}}$$

$$\tan(-\beta)=-\tan\beta$$

$$\tan(\alpha-\beta)=\tan\big(\alpha+(-\beta)\big)$$
$$=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\cdot\tan(-\beta)}$$

$$\boxed{\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}}$$

DOUBLE ANGLE FORMULAE

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$$

$$\sin(2\theta)=\sin(\theta+\theta)=\sin\theta\cdot\cos\theta+\cos\theta\cdot\sin\theta$$

$$\boxed{\sin(2\theta)=2\cdot\sin\theta\cdot\cos\theta}$$

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$$

$$\cos(2\theta)=\cos(\theta+\theta)=\cos\theta\cdot\cos\theta-\sin\theta\cdot\sin\theta$$

$$\boxed{\cos(2\theta)=\cos^2\theta-\sin^2\theta}$$

$$\cos(2\theta)=(1-\sin^2\theta)-\sin^2\theta \quad \boxed{=1-2\cdot\sin^2\theta}$$

$$\cos(2\theta)=\cos^2\theta-(1-\cos^2\theta) \quad \boxed{=2\cdot\cos^2\theta-1}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\tan(2\theta)=\tan(\theta+\theta)=\frac{\tan\theta+\tan\theta}{1-\tan\theta\cdot\tan\theta}$$

$$\boxed{\tan(2\theta)=\frac{2\cdot\tan\theta}{1-\ \tan^2\theta}} \qquad \cdot\left(\frac{\cos^2\theta}{\cos^2\theta}\right)=$$

$$\tan(2\theta)=\frac{\sin(2\theta)}{\cos(2\theta)} \qquad \boxed{=\frac{2\cdot\sin\theta\cdot\cos\theta}{\cos^2\theta-\sin^2\theta}}$$

HALF ANGLE FORMULAE

$$\cos(2\theta) = 1 - 2\cdot\sin^2\theta \qquad \cos(2\theta) = 2\cdot\cos^2\theta - 1$$

$$2\cdot\sin^2\theta = 1 - \cos(2\theta) \qquad 2\cdot\cos^2\theta = 1 + \cos(2\theta)$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}} \qquad \cos\theta = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}$$

$$\frac{\emptyset}{2} = \theta$$

$$\boxed{\sin\frac{\emptyset}{2} = \pm\sqrt{\frac{1-\cos(\emptyset)}{2}}} \qquad \boxed{\cos\frac{\emptyset}{2} = \pm\sqrt{\frac{1+\cos(\emptyset)}{2}}}$$

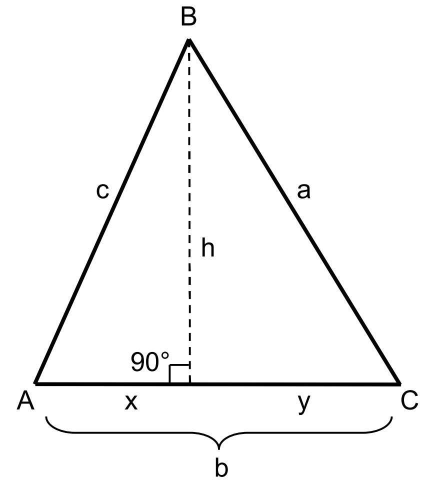
$$\tan\frac{\emptyset}{2} = \frac{\sin\frac{\emptyset}{2}}{\cos\frac{\emptyset}{2}} = \frac{\pm\sqrt{\frac{1-\cos(\emptyset)}{2}}}{\pm\sqrt{\frac{1+\cos(\emptyset)}{2}}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}}\cdot\frac{\sqrt{1+\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}} = \frac{\sqrt{1-\cos^2(\emptyset)}}{1+\cos(\emptyset)} \quad \boxed{= \frac{\sin(\emptyset)}{1+\cos(\emptyset)}}$$

$$\tan\frac{\emptyset}{2} = \frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1+\cos(\emptyset)}}\cdot\frac{\sqrt{1-\cos(\emptyset)}}{\sqrt{1-\cos(\emptyset)}} = \frac{1-\cos(\emptyset)}{\sqrt{1-\cos^2(\emptyset)}} \quad \boxed{= \frac{1-\cos(\emptyset)}{\sin(\emptyset)}}$$

$$\boxed{\tan\frac{\emptyset}{2} = \pm\sqrt{\frac{1-\cos(\emptyset)}{1+\cos(\emptyset)}}}$$

LAW of SINES DERIVATION



Equation 1 $h = c \cdot \sin A$

Equation 2 $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange…

Equation 3 $\dfrac{a}{\sin A} = \dfrac{c}{\sin C}$

Note: $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ and $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ are derived similarly.

Law of Sines… $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$ or $\boxed{\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}}$

LAW of COSINES DERIVATION

Equation 4 $h^2 = a^2 - y^2 = a^2 - (b - x)^2$

Equation 5 $h^2 = c^2 - x^2$

Equate the right sides of equations 4 and 5 and rearrange…

$a^2 - (b - x)^2 = c^2 - x^2$

Equation 6 $a^2 = b^2 + c^2 - 2bx$

Equation 7 $x = c \cdot \cos A$

Substitute for x from equation 7 into equation 6…

Equation 8 $a^2 = b^2 + c^2 - 2bc \cdot \cos A$

Note: equations for b^2 and c^2 are derived similarly.

Law of Cosines…

alternate versions

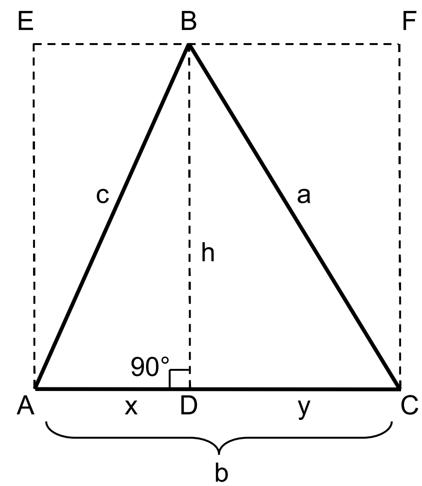
$\boxed{\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}}$ or $\boxed{a^2 = b^2 + c^2 - 2bc \cdot \cos A}$

$\boxed{\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}}$ or $\boxed{b^2 = a^2 + c^2 - 2ac \cdot \cos B}$

$\boxed{\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}}$ or $\boxed{c^2 = a^2 + b^2 - 2ab \cdot \cos C}$

Area ABD = Area ABE = $\frac{x \cdot h}{2}$

Area CBD = Area CBF = $\frac{y \cdot h}{2}$



Therefore…

Area of Triangle ABC…

$$Area = \frac{b \cdot h}{2}$$

$$h = a \cdot \sin C$$

Therefore…

Area of Triangle ABC…

$$Area = \frac{a \cdot b \cdot \sin C}{2}$$

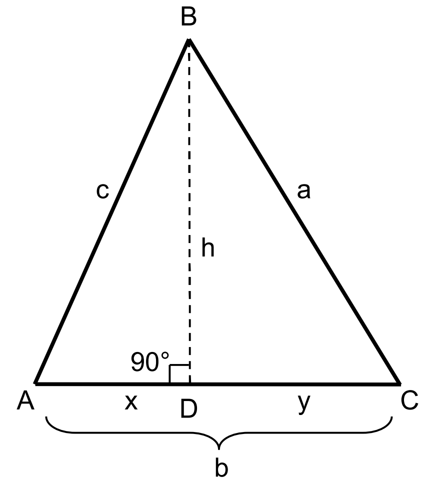
$$b = \frac{a \cdot \sin B}{\sin A}$$

Therefore…

Area of Triangle ABC…

$$Area = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$

HERON'S FORMULA DERIVATION



Equation 1 $$Area = \frac{b \cdot h}{2}$$

From triangle ABD… $$x^2 + h^2 = c^2$$

Equation 2 $$x^2 = c^2 - h^2$$

Equation 3 $$x = \sqrt{c^2 - h^2}$$

From triangle CBD… $$(b - x)^2 + h^2 = a^2$$

$$(b - x)^2 = a^2 - h^2$$

Equation 4 $$b^2 - 2bx + x^2 = a^2 - h^2$$

Substitute x and x^2 from Equations 3 and 2 into Equation 4 …

$$b^2 - 2b\sqrt{c^2 - h^2} + (c^2 - h^2) = a^2 - h^2$$

$$b^2 + c^2 - a^2 = 2b\sqrt{c^2 - h^2}$$

Square both sides … $$(b^2 + c^2 - a^2)^2 = 4b^2(c^2 - h^2)$$

$$\frac{(b^2+c^2-a^2)^2}{4b^2} = c^2 - h^2$$

$$h^2 = c^2 - \frac{(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{4b^2c^2-(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{(2bc)^2-(b^2+c^2-a^2)^2}{4b^2}$$

$$h^2 = \frac{[2bc+(b^2+c^2-a^2)]\cdot[2bc-(b^2+c^2-a^2)]}{4b^2}$$

$$h^2 = \frac{[2bc+b^2+c^2-a^2]\cdot[2bc-b^2-c^2+a^2]}{4b^2}$$

$$h^2 = \frac{[(b^2+2bc+\ ^2)-a^2]\cdot[a^2-(b^2-2bc+c^2)]}{4b^2}$$

$$h^2 = \frac{[(b+c)^2 - a^2]\cdot[a^2 - (b-c)^2]}{4b^2}$$

$$h^2 = \frac{[(b+c)+a]\cdot[(b+c)-a]\cdot[a+(b-c)]\cdot[a-(b-c)]}{4b^2}$$

$$h^2 = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{4b^2}$$

$$h^2 = \frac{(a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)}{4b^2}$$

Since $P = a + b + c$ …

$$h^2 = \frac{P(P-2a)(P-2b)(P-2c)}{4b^2}$$

<u>Equation 5</u>

$$h = \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

Substitute h from Equation 5 into Equation 1 …

$$Area = \frac{1}{2} b \frac{\sqrt{P(P-2a)(P-2b)(P-2c)}}{2b}$$

$$Area = \frac{1}{4}\sqrt{P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\frac{1}{16}P(P-2a)(P-2b)(P-2c)}$$

$$Area = \sqrt{\left(\frac{P}{2}\right)\left(\frac{P-2a}{2}\right)\left(\frac{P-2b}{2}\right)\left(\frac{P-2c}{2}\right)}$$

$$Area = \sqrt{\frac{P}{2}\left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(\frac{P}{2} - c\right)}$$

Thus...

Given the three sides of a triangle (a, b, and c) …

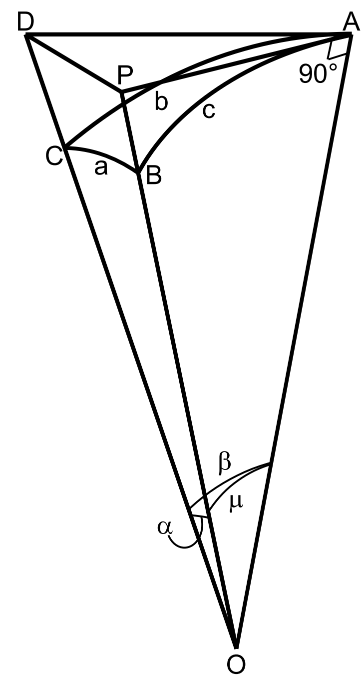
The area of the triangle is:

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$

Where the semi perimeter is:

$$s = P/2 = (a+b+c)/2$$

LAW of COSINES DERIVATION



Sides AD and AP are tangent at A such that…

DAP = spherical angle A
DAO = 90°
PAO = 90°

In plane triangle DPO…

Equation 1 $$DP^2 = PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos\alpha$$

In plane triangle ADP…

Equation 2 $$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In right triangle APO…

Equation 3 $$AO^2 = PO^2 - AP^2$$

In right triangle ADO…

Equation 4 $$AO^2 = DO^2 - AD^2$$

Equate the right sides of equations 1 and 2 and rearrange…

$$PO^2 + DO^2 - 2 \cdot PO \cdot DO \cdot \cos\alpha = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

Equation 5 $$PO^2 + DO^2 - AD^2 - AP^2 = 2 \cdot PO \cdot DO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

Add equations 3 and 4 and rearrange…

$$AO^2 + AO^2 = PO^2 - AP^2 + DO^2 - AD^2$$

Equation 6 $$2 \cdot AO^2 = PO^2 + DO^2 - AD^2 - AP^2$$

Equate the left side of equation 6 to the right side of equation 5 and rearrange…

$$2 \cdot AO^2 = 2 \cdot PO \cdot DO \cdot \cos\alpha - 2 \cdot AD \cdot AP \cdot \cos A$$
$$AO^2 + AD \cdot AP \cdot \cos A = PO \cdot DO \cdot \cos\alpha$$

$$\cos\alpha = \frac{AO \cdot AO}{PO \cdot DO} + \frac{AP \cdot AD}{PO \cdot DO}\cos A \qquad = \cos\mu \cdot \cos\beta + \sin\mu \cdot \sin\beta \cdot \cos A$$

note… $\alpha = a$, $\beta = b$, $\mu = c$

Law of Cosines…

$$\boxed{\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A}$$

$$\boxed{\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B}$$

$$\boxed{\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C}$$

alternate versions

LAW of SINES DERIVATION

<u>Equation 7</u> $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$

<u>Equation 8</u> $\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$

<u>Equation 9</u> $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$

(Equations 7–9:) by Law of Cosines

Rearrange equations 7, 8 and 9…

<u>Equation 10</u> $\cos a - \cos b \cdot \cos c = \cos A \cdot \sin b \cdot \sin c$ (from 7)

<u>Equation 11</u> $\cos b - \cos a \cdot \cos c = \cos B \cdot \sin a \cdot \sin c$ (from 8)

<u>Equation 12</u> $\cos c - \cos a \cdot \cos b = \cos C \cdot \sin a \cdot \sin b$ (from 9)

Using equation 10, square both sides…

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \cos^2 A \cdot \sin^2 b \cdot \sin^2 c$$

Apply the trig identity $\cos^2 \theta = 1 - \sin^2 \theta$ …

$$\left(1 - \sin^2 a\right) - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \left(1 - \sin^2 b\right) \cdot \left(1 - \sin^2 c\right) = \left(1 - \sin^2 A\right) \cdot \sin^2 b \cdot \sin^2 c$$

$$1 - \sin^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + 1 - \sin^2 c - \sin^2 b + \sin^2 b \cdot \sin^2 c = \sin^2 b \cdot \sin^2 c - \sin^2 A \cdot \sin^2 b \cdot \sin^2 c$$

<u>Equation 13</u> $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 A \cdot \sin^2 b \cdot \sin^2 c$

Using equation 11 yields…

<u>Equation 14</u> $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 B \cdot \sin^2 a \cdot \sin^2 c$

Using equation 12 yields…

<u>Equation 15</u> $2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 C \cdot \sin^2 a \cdot \sin^2 b$

Equate the right sides of equations 13, 14 and 15 and rearrange…

<u>Law of Sines…</u>

$$\boxed{\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}}$$

LAW of COTANGENTS DERIVATION

Equation 16 $\sin b \cdot \sin A = \sin a \cdot \sin B$ … by Law of Sines

Equation 17 $\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$

Equation 18 $\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$

Equation 19 $\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$

(Equations 17–19: by Law of Cosines)

Substitute for $\cos b$ from equation 18 into equation 17…

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity $\sin^2 \theta = 1 - \cos^2 \theta$ …

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

Rearrange and divide both sides with equation 16…

$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

Law of Cotangents…

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

Or …

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$

Spherical Excess…

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

where

$$s = \frac{a+b+c}{2}$$